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A space cut-off approach to scattering involving Coulomb-like potentials

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Abstract. A prescription is given by which the distorted waves and the on-energy-shell T matrix for particles interacting via Coulomb-like potentials can be determined from the corresponding screened distorted waves and on-energy-shell screened T matrix.

1. Introduction

The cut-off or screened approach provides a method for studying scattering involving Coulomb-like potentials via the standard formalism of short-range scattering theory. In this approach the Coulomb potentials are replaced by screened Coulomb potentials which are chosen to be of short range and to reduce to the Coulomb potential when the cut-off is removed. Since the screened system of particles involves only short-range potentials the standard formalism of short-range potential scattering theory is valid. For example in the case of three-particle scattering the Faddeev equations (Faddeev 1965) can be used, at least in principle, to determine the distorted waves and T matrix of the cut-off scattering theory. The key question is whether the distorted waves and T matrix of the screened scattering theory converge to the corresponding distorted waves and T matrix of the Coulomb-like scattering theory as the cut-off is removed.

In this paper it will be shown that if the distorted waves and the T matrix of the cut-off stationary scattering theory are first multiplied by certain momentum and cut-off dependent phase factors then the resulting expressions will converge to the corresponding distorted waves and T matrix of the Coulomb-like scattering theory.

The physical system will be assumed to consist of N spinless distinct particles interacting via Coulomb-like potentials. Thus the system will be described by

$$H = H_0 + V, \qquad H_0 = -\sum_{i=1}^{N} (2m_i)^{-1}\nabla_i^2,$$
$$V = \sum_{i<i'} V_{ii'}(\boldsymbol{x}_i - \boldsymbol{x}_{i'}) \tag{1.1}$$

where H acts in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{3N})$ and the two-body potentials are assumed to be Coulomb-like, ie

$$V_{ii'}(\boldsymbol{x}) = V_{ii'}^{(s)}(\boldsymbol{x}) + \hat{e}_i\hat{e}_{i'}|\boldsymbol{x}|^{-1} \tag{1.2}$$

where $\hat{e}_i$ denotes the charge of the ith particle and

$$V_{ii'}^{(s)}(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{-2-\epsilon_0}), \qquad \epsilon_0 > 0. \tag{1.3}$$

In order to relate the Coulomb-like scattering theory to the corresponding cut-off scattering theory we will make use of the time-dependent formulation of scattering involving Coulomb potentials first given by Dollard (1963, 1964). It was shown that there exist 'modified' or 'renormalized' (Prugovečki 1971b) wave operators defined by

$$\Omega_{\pm}^{(\gamma)} = s-\lim_{t\to\pm\infty} \exp(\mathrm{i}Ht)U^{(\gamma)}(t)P^{(\gamma)} \tag{1.4}$$

with $P^{(\gamma)}$ denoting the projection onto the channel subspace $\mathscr{H}^{(\gamma)}$. The unitary operators $U^{(\gamma)}(t)$ take the following form (Dollard 1963, 1964):

$$U^{(\gamma)}(t) = \exp\left[-\mathrm{i}H^{(\gamma)}t - \mathrm{i}\epsilon(t)\sum_{j<k}\frac{M_jM_ke_je_k}{|M_j\nabla_k - M_k\nabla_j|}\ln\left(\frac{2|t||M_j\nabla_k - M_k\nabla_j|^2}{M_jM_k(M_j+M_k)}\right)\right],$$

$$\epsilon(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases} \tag{1.5}$$

with M_j and e_j denoting the total mass and charge of the jth fragment of the n-fragment system and ∇_j representing the gradient with respect to the coordinates of the centre-of-mass of the jth fragment.

In § 2 the relationship between the Coulomb-like distorted waves and the corresponding distorted waves for the cut-off Coulomb-like potentials is derived. The derivation of this relationship is a generalization of results obtained previously (Prugovečki and Zorbas 1973b) which involved screening only between fragments. Section 2 will be concerned with the case when screening is introduced between the individual particles.

In § 3 a prescription is given which allows one to recover the T matrix for Coulomb-like interactions from the corresponding T matrix for cut-off Coulomb-like interactions.

The paper concludes with a general discussion of the applicability of the cut-off approach to potential scattering.

2. The cut-off approach to the Coulomb-like distorted waves

In this section the relationship between the Coulomb-like distorted waves and the corresponding cut-off distorted waves will be derived. This will be accomplished by generalizing a technique first used by Dollard (1966, 1968) for proving that the two-particle wave operators for adiabatically switched or screened Coulomb-like potentials converge to zero when the switching or screening is removed.

There are several possible ways to introduce screening within a system of particles. It will prove convenient when deriving the T matrix formalism in § 3 to distinguish two cut-off scattering theories. When all Coulomb potentials are screened the screened system of particles will be assumed to interact via the following hamiltonian:

$$H_1 = H_0 + V_{(\gamma)}^{(s)} + \sum_{\substack{\iota<\iota' \\ \iota,\iota' \text{ members of} \\ \text{the same fragment}}} \hat{e}_\iota\hat{e}_{\iota'}g_{\iota\iota'}^{R}(|\boldsymbol{x}_\iota - \boldsymbol{x}_{\iota'}|)|\boldsymbol{x}_\iota - \boldsymbol{x}_{\iota'}|^{-1} + V_{(\gamma)}(R)$$

$$V_{(\gamma)}(R) = V^{(s)} + \sum_{\substack{\iota<\iota' \\ \iota,\iota' \text{ members of} \\ \text{different fragments}}} \hat{e}_\iota\hat{e}_{\iota'}g_{\iota\iota'}^{R}(|\boldsymbol{x}_\iota - \boldsymbol{x}_{\iota'}|)|\boldsymbol{x}_\iota - \boldsymbol{x}_{\iota'}|^{-1} \tag{2.1}$$

with $V^{(s)}$ denoting all short-range potentials which are not contained in the channel hamiltonian $H_1^{(\gamma)} = H_1 - V_{(\gamma)}(R)$. For the case when the Coulomb potentials acting within the fragments of the channel γ are not screened the system of particles will be described by the following hamiltonian:

$$H_2 = H_0 + V_{(\gamma)}^{(s)} + \sum_{\substack{i<i' \\ i,i' \text{ members of} \\ \text{the same fragment}}} \hat{e}_i \hat{e}_{i'} |\boldsymbol{x}_i - \boldsymbol{x}_{i'}|^{-1} + V_{(\gamma)}(R) \tag{2.2}$$

with $V_{(\gamma)}(R)$ given in (2.1). The screening functions $g_{ii'}^R$ will be assumed to be Schwartz functions for each i, i' which satisfy the following requirements:

$$\lim_{R \to +\infty} g_{ii'}^R(\xi) = 1$$

$$\left| \xi^\nu \frac{\mathrm{d}^\nu}{\mathrm{d}\xi^\nu} g_{ii'}^R(\xi) \right| \leqslant C_\nu. \qquad \nu = 0, 1, \ldots, \text{with } C_0 = 1. \tag{2.3}$$

If the system of particles involves compound fragments we will also require the $g_{ii'}^R$ for each i, i' to satisfy the following condition:

$$|g_{ii'}^R(\xi)\eta - g_{ii'}^R(\eta)\xi| \leqslant C|\eta - \xi|. \tag{2.4}$$

One can easily see that the conditions (2.3) and (2.4) are satisfied by the exponential screening function

$$g_{ii'}^R(\xi) = \exp(-R^{-1}|\xi|). \tag{2.5}$$

We note that the condition (7.3)† imposed on the screening functions by Prugovečki and Zorbas (1973b) only applies to the exponential screening function as stated. This can be rectified by replacing the requirement (7.3) by the condition contained in (2.3) of this paper. With this replacement all the results concerning cut-offs contained in Prugovečki and Zorbas (1973b) are valid as stated for any screening function which satisfies (2.3).

We will now define cut-off dependent operators which will provide the link between the renormalized and cut-off wave operators. Thus for $|t| > t_0 > 1$ we define the following operators:

$$\tilde{\Omega}_l^{(\gamma)}(t; R) = \exp(iH_l t) U_l^{(\gamma)}(t) P_l^{(\gamma)}, \qquad l = 1, 2 \tag{2.6}$$

where H_1 and H_2 are defined in (2.1) and (2.2) respectively and $P_l^{(\gamma)}, l = 1, 2$ are the projection operators which project onto the γ channel subspace $\mathscr{H}_l^{(\gamma)}, l = 1, 2$. The unitary operators $U_l^{(\gamma)}(t)$ are given by

$$U_l^{(\gamma)}(t) = \exp(-\mathrm{i}K_l^{(\gamma)}(t)), \qquad l = 1, 2 \tag{2.7}$$

with

$$K_l^{(\gamma)}(t) = H_l^{(\gamma)} t + \epsilon(t) \sum_{j<k} \frac{M_j M_k e_j e_k}{|M_j \nabla_k - M_k \nabla_j|} \times \left[\int_{t_0}^{|t|} \mathrm{d}s\, s^{-1} g_{jk}^R \left(\frac{s}{M_j M_k} |M_j \nabla_k - M_k \nabla_j| \right) + \ln \left(\frac{2t_0 |M_j \nabla_k - M_k \nabla_j|^2}{M_j M_k (M_j + M_k)} \right) \right] \tag{2.8}$$

chosen so as to reduce to (1.5) when $R = \infty$.

We will now require the following lemmas which are generalizations of the time-dependent results proven by Dollard (1966) for the case of two particles interacting via Yukawa potentials.

† The first requirement in (7.3) (Prugovečki and Zorbas 1973b) should read $|(\xi R^{-1})^\nu g_{jk}^R(\xi)| \leqslant C_\nu$

Lemma 2.1. Suppose that the screening functions $g^R_{kk'}$ for each k, k' are Schwartz functions which satisfy the conditions (2.3) and (2.4). In addition assume that the bound state wavefunction $\psi^l_i(x^1_i, \ldots, x^{n_i-1}_i)$ for each fragment i of the channel γ with $x^1_i, \ldots, x^{n_i-1}_i$ denoting the internal coordinates of the fragment satisfies for some $\beta > 0$

$$\int \mathrm{d}x^1_i \ldots \mathrm{d}x^{n_i-1}_i |\psi^l_i(x^1_i, \ldots, x^{n_i-1}_i)|^2 |x^j_i|^\beta < \tilde{C}, \qquad j = 1, \ldots, n_i - 1 \quad (2.9)$$

where $\tilde{C}$ is a constant which is independent of R. Then the strong limits

$$s - \lim_{t \to \pm\infty} \tilde{\Omega}^{(\gamma)}_l(t; R) = \tilde{\Omega}^{(\gamma)}_{l\pm}(R) \qquad (2.10)$$

exist and the convergence is uniform in $R > 0$.

In order to prove (2.10) it is sufficient to show that for some $t_0 > 1$

$$\pm \int_{\pm t_0}^{\pm\infty} \mathrm{d}t \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\tilde{\Omega}^{(\gamma)}_l(t; R))\Psi \right\|$$
$$= \pm \int_{\pm t_0}^{\pm\infty} \mathrm{d}t \left\| \left(V^{(s)} + \sum_{i < i'} \frac{\hat{e}_i \hat{e}_{i'} g^R_{ii'}(|x_i - x_{i'}|)}{|x_i - x_{i'}|} \right.\right.$$
$$\left.\left. - \sum_{j<k} \frac{M_j M_k e_j e_k g^R_{jk}[(|t|/M_j M_k)|M_j \nabla_k - M_k \nabla_j|]}{|t|\,|M_j \nabla_k - M_k \nabla_j|} \right) U^{(\gamma)}_l(t)\Psi \right\| \leqslant C' \qquad (2.11)$$

for all Ψ in a dense subset of $\mathscr{H}^{(\gamma)}$ where C' is a constant which is independent of R. The proof of the inequality (2.11) is analogous to the proof of the existence of the renormalized wave operators (Dollard 1963, theorem 4, p 148). The technical conditions (2.3) and (2.4) placed on the screening functions $g^R_{kk'}$ are required in order to obtain an R independent bound. Since the detailed proof of the above lemma is somewhat involved it will be given elsewhere (Zorbas 1974).

The condition (2.9) in the case $l = 2$ can be shown to be satisfied if each potential in the ith fragment is assumed to be a C^∞ function on an open subset of $\mathbb{R}^3$ whose complement is of measure zero (Hunziker 1966a, 1966b, theorem 4). In order for (2.9) to be satisfied in the case $l = 1$ we must also require that the bound can be made independent of R.

The proofs of the following two lemmas are analogous to the proofs of lemma 6 and lemma 7 respectively of Prugovečki and Zorbas (1973b) and thus will be omitted.

Lemma 2.2. Suppose the projectors $P^{(\gamma)}_l$ in (2.6) converge strongly to the projector $P^{(\gamma)}$ in (1.4), ie

$$s - \lim_{R \to +\infty} P^{(\gamma)}_l = P^{(\gamma)}. \qquad (2.12)$$

Then we have

$$s - \lim_{R \to +\infty} \tilde{\Omega}^{(\gamma)}_{l\pm}(R) = \Omega^{(\gamma)}_\pm \qquad (2.13)$$

and

$$\omega - \lim_{R \to +\infty} \tilde{\Omega}^{(\gamma)*}_{l\pm}(R) = \Omega^{(\gamma)*}_\pm. \qquad (2.14)$$

We note that for screened scattering theories whose hamiltonian is given by (2.2), ie $l = 2$, condition (2.12) is immediately valid. For $l = 1$ condition (2.12) is a question of bound state perturbation theory. Indeed if corresponding to each eigenfunction ψ_i of the ith fragment there exists an eigenfunction ψ_i^1 of the cut-off hamiltonian for the ith fragment which converges strongly to ψ_i as the cut-off is removed then (2.12) will be true.

Lemma 2.3. The wave operators

$$\Omega_{l\pm}^{(\gamma)}(R) = s - \lim_{t\to\pm\infty} \exp(\mathrm{i}H_l t)\exp(-\mathrm{i}H_l^{(\gamma)}t)P_l^{(\gamma)} \tag{2.15}$$

exist and

$$\Omega_{l\pm}^{(\gamma)}(R)\exp(\mp\mathrm{i}\Lambda^{(\gamma)}(R)) = \tilde{\Omega}_{l\pm}^{(\gamma)}(R) \tag{2.16}$$

for each $0 < R < \infty$ where

$$(\Lambda^{(\gamma)}(R)\Psi)(p_\gamma;\omega_\gamma)$$

$$= \sum_{j<k} \frac{M_j M_k e_j e_k}{|M_j p_k - M_k p_j|}\left[\int_{t_0}^{+\infty} \mathrm{d}s\, s^{-1} g_{jk}^R\left(\frac{s}{M_j M_k}|M_j p_k - M_k p_j|\right)\right.$$

$$\left. + \ln\left(\frac{2t_0|M_j p_k - M_k p_j|^2}{M_j M_k(M_j + M_k)}\right)\right]\Psi(p_\gamma;\omega_\gamma) \tag{2.17}$$

with p_γ collectively denoting the centre-of-mass momentum variables and ω_γ the internal degrees of freedom variables.

The following theorem will yield the representation of the renormalized wave operators in terms of the cut-off wave operators.

Theorem 2.1. Consider the cut-off scattering theories described by (2.1) and (2.2) with the Schwartz functions $g_{ii'}^R$ for each i, i' satisfying the conditions (2.3) and (2.4). Furthermore assume that the requirements (2.9) and (2.12) are satisfied. Then

$$s - \lim_{R\to+\infty} \Omega_{l\pm}^{(\gamma)}(R)\exp(\mp\mathrm{i}\Lambda^{(\gamma)}(R)) = \Omega_{\pm}^{(\gamma)}, \qquad l = 1, 2 \tag{2.18}$$

where $\Lambda^{(\gamma)}(R)$ is given by (2.17).

Proof. The above relations follow immediately from (2.13) and (2.16).

In order to derive the relationship between the Coulomb-like distorted waves and the cut-off Coulomb-like distorted waves we must express (2.18) in terms of eigenfunctions. This can be done by applying the extended Hilbert space formalism (Prugovečki 1973b) to the relations (2.18) for the case $l = 2$. Since the screened theory (2.2) does not involve cut-offs in the channel hamiltonian the corresponding momentum eigenfunctions $\Phi_{p_\gamma,\omega_\gamma}^{(\gamma)}$ do not depend on the cut-off parameter R. Thus denoting

$$(\Omega_{2\pm}^{(\gamma)}(R)^{*\dagger}\Phi_{p_\gamma,\omega_\gamma}^{(\gamma)})(x;R) \equiv \Phi_{p_\gamma,\omega_\gamma}^{(\gamma)\pm}(x;R), \qquad (\Omega_{\pm}^{(\gamma)*\dagger}\Phi_{p_\gamma,\omega_\gamma}^{(\gamma)})(x) \equiv \Phi_{p_\gamma,\omega_\gamma}^{(\gamma)\pm}(x) \tag{2.19}$$

as the cut-off distorted waves and the Coulomb-like distorted waves respectively with a dagger denoting the bra-adjoint (Prugovečki 1973b) we arrive at the following relations:

$$\Phi_{p_\gamma,\omega_\gamma}^{(\gamma)\pm}(x) = \lim_{R\to+\infty} \exp(\mp\mathrm{i}\Lambda^{(\gamma)}(R))\Phi_{p_\gamma,\omega_\gamma}^{(\gamma)\pm}(x;R). \tag{2.20}$$

The pointwise interpretation of the cut-off limit $R \to +\infty$ depends on the assumption that the limit exists. If this limit does not exist in a pointwise sense then the relations

(2.20) are to be understood in the sense of their derivation from the Hilbert space relations (2.18). For a further discussion of the cut-off approach to the Coulomb distorted waves we refer the reader to Prugovečki and Zorbas (1973b).

3. A cut-off formalism for the T matrix

The relationship between the on-energy-shell T matrix for Coulomb scattering denoted by $\langle p_\beta, \omega_\beta | T_{\alpha\beta} | p'_\alpha, \omega'_\alpha \rangle_{E^{(\beta)}=E^{(\alpha)'}}$ (Prugovečki and Zorbas 1973b, equation (4.18)) and the corresponding on-energy-shell screened T matrix will be derived in this section. In order to simplify the derivation of this relationship we will distinguish two cases depending on whether the scattering process does or does not satisfy the following condition (to be denoted 3.A).

The scattering process which is associated with the original scattering system by neglecting all uncharged particles is such that it does not involve a re-arrangement of the charged particles.

In the case of collisions for which condition (3.A) is satisfied the following relationship will be shown to hold:

$$\begin{aligned}\langle p_\beta, \omega_\beta | T_{\alpha\beta} | p'_\alpha, \omega'_\alpha \rangle_{E^{(\beta)}=E^{(\alpha)'}} &= \frac{\delta_{\alpha\beta}}{2\pi \mathrm{i}} \langle p_\beta, \omega_\beta | p'_\alpha, \omega'_\alpha \rangle_{E^{(\beta)}=E^{(\alpha)'}} \\ &+ \lim_{R\to+\infty} [\exp(\mathrm{i}\Lambda^{(\alpha)}(R) + \mathrm{i}\Lambda^{(\beta)}(R)) \langle p_\beta, \omega_\beta | V_{(\beta)}(R) \Omega^{(\alpha)}_{2-}(R) | p'_\alpha, \omega'_\alpha \rangle]_{E^{(\beta)}=E^{(\alpha)'}} \end{aligned} \tag{3.1}$$

where the on-energy-shell restriction is denoted by $E^{(\beta)} = E^{(\alpha)'}$ with

$$E^{(\gamma)} = \sum_{j=1}^{n} (2M_j)^{-1} p_j^2 + E_{(\gamma)} \tag{3.2}$$

and $E_{(\gamma)}$ denoting the internal energy of the fragments. In the case of collision processes for which condition (3.A) is not satisfied analogous relations (3.14) to those above will be shown to hold.

The derivation of (3.1) and (3.14) will be based on the T operator $T_{\alpha\beta}$ which is given by

$$T_{\alpha\beta} = \frac{1}{2\pi \mathrm{i}} (\delta_{\alpha\beta} P^{(\beta)} - \Omega^{(\beta)*}_{+} \Omega^{(\alpha)}_{-}).$$

In order to simplify the comparison with the results of Prugovečki and Zorbas (1973b) we will restrict $T_{\alpha\beta}$ as follows

$$T^{\Delta}_{\alpha\beta} \equiv P^{(\beta)\Delta_\beta} T_{\alpha\beta} P^{(\alpha)\Delta_\alpha} = \frac{1}{2\pi \mathrm{i}} (\delta_{\alpha\beta} P^{(\beta)\Delta_\beta} - \Omega^{(\beta)\Delta^*_\beta}_{+} \Omega^{(\alpha)\Delta_\alpha}_{-}) \tag{3.3}$$

where $\Omega^{(\gamma)\Delta_\gamma}_{\pm} = \Omega^{(\gamma)}_{\pm} P^{(\gamma)\Delta_\gamma}$ with $P^{(\gamma)\Delta_\gamma}$ the projection operator which projects onto the subspace of functions of $\mathscr{H}^{(\gamma)}$ whose support is contained in the compact subset Δ_γ of momentum space. Due to the localized nature of the momentum eigenfunctions in momentum space the above restriction will not affect the derivation of (3.1) and (3.14) and will be in accord with the derivation of the on-energy-shell Coulomb T matrix (Prugovečki and Zorbas 1973b). We also note that the results of § 2 are valid when restricted to $P^{(\gamma)\Delta_\gamma} \mathscr{H}^{(\gamma)}$.

By lemma (2.2) we can express $T^{\Delta}_{\alpha\beta}$ as follows

$$T^{\Delta}_{\alpha\beta} = \frac{1}{2\pi i}\omega - \lim_{R\to+\infty} (\delta_{\alpha\beta}P^{(\beta)\Delta_\beta} - \tilde{\Omega}^{(\beta)\Delta_\beta}_{l+}(R)^*\tilde{\Omega}^{(\alpha)\Delta_\alpha}_{l-}(R)), \qquad l = 1, 2. \tag{3.4}$$

We will now require the following lemma which will enable us to relate $T^{\Delta}_{\alpha\beta}$ to the cut-off T operators.

Lemma 3.1. Suppose that the conditions (2.12) are valid and that the wave operators of the cut-off scattering theories $\Omega^{(\gamma)\Delta_\gamma}_{l\pm}(R)$ satisfy for each $0 < R < \infty$

$$\Omega^{(\beta)\Delta_\beta}_{l-}(R)^*\Omega^{(\alpha)\Delta_\alpha}_{l-}(R) = \delta_{\alpha\beta}P^{(\beta)\Delta_\beta}_l. \tag{3.5}$$

Furthermore assume that

$$\omega - \lim_{R\to+\infty} \exp(\pm 2i\Lambda^{(\alpha)}(R)) = 0. \tag{3.6}$$

Then

$$\omega - \lim_{R\to+\infty} \exp(i\Lambda^{(\beta)}(R))\Omega^{(\beta)\Delta_\beta}_{l-}(R)^*\Omega^{(\alpha)\Delta_\alpha}_{l-}(R)\exp(i\Lambda^{(\alpha)}(R)) = 0 \tag{3.7}$$

for all channels α and β.

Proof. The case $\alpha \neq \beta$ follows immediately by (3.5). For $\alpha = \beta$ we can write for each $0 < R < \infty$

$$\exp(i\Lambda^{(\alpha)}(R))\Omega^{(\alpha)\Delta_\alpha}_{l-}(R)^*\Omega^{(\alpha)\Delta_\alpha}_{l-}\exp(i\Lambda^{(\alpha)}(R)) = \exp(2i\Lambda^{(\alpha)}(R))P^{(\alpha)\Delta_\alpha}_l.$$

Since $P^{(\alpha)\Delta_\alpha}_l$ converges strongly as $R \to +\infty$ and $\exp(2i\Lambda^{(\alpha)}(R))$ converges weakly to zero as $R \to +\infty$ we conclude that (3.7) is true.

The relations (3.5) for $\alpha = \beta$ are valid since $\Omega^{(\alpha)\Delta_\alpha}_{l-}(R)$ are partial isometries for each $0 < R < \infty$ (Prugovečki 1971a, theorem (2.2), lemma (2.1), p 418). The case $\alpha \neq \beta$ follows from the uniqueness of the asymptotic states (Prugovečki 1971a, theorem (8.1), p 583). Due to the rapidly oscillating behaviour of the functions for large R it is easy to see that condition (3.6) is true.

Using the relations (3.7) we can rewrite (3.4) as follows

$$T^{\Delta}_{\alpha\beta} = \omega - \lim_{R\to+\infty}\left(\frac{\delta_{\alpha\beta}}{2\pi i}P^{\beta\Delta_\beta} + \exp(i\Lambda^{(\beta)}(R))T^{\Delta}_{\alpha\beta;l}(R)\exp(i\Lambda^{(\alpha)}(R))\right) \tag{3.8}$$

where

$$T^{\Delta}_{\alpha\beta;l}(R) = \frac{1}{2\pi i}(\Omega^{(\beta)\Delta_\beta}_{l-}(R) - \Omega^{\beta\Delta_\beta}_{l+}(R))^*\Omega^{(\alpha)\Delta_\alpha}_{l-}(R) \tag{3.9}$$

corresponds to the T operator of the cut-off scattering theories. Thus (3.8) yields the relationship between the T operator for Coulomb potentials and the corresponding screened T operators.

In order to obtain the standard expression for the on-energy-shell screened T matrix we must express the screened T operators (3.9) in terms of Riemann–Stieltjes integrals (Prugovečki 1971a)

$$\Omega^{(\beta)\Delta_\beta}_{l\pm}(R) = s - \lim_{\epsilon\to+0}\int_{-\infty}^{+\infty}\frac{\pm i\epsilon}{H_l - \lambda \pm i\epsilon}\,d_\lambda E^{H(\beta)}_\lambda P^{(\beta)\Delta_\beta}_l, \qquad l = 1, 2.$$

Under the assumption that the following intertwining properties hold

$$\frac{\epsilon}{(H_l-\lambda)^2+\epsilon^2}\Omega_{l-}^{(\alpha)\Delta_\alpha}(R) = \Omega_{l-}^{(\alpha)\Delta_\alpha}(R)\frac{\epsilon}{(H_l^{(\alpha)}-\lambda)^2+\epsilon^2}, \qquad \lambda\in\mathbb{R}^1 \tag{3.10}$$

and applying lemma 2.1 of Prugovecki (1973a) we obtain for $T^{\Delta}_{\alpha\beta;l}(R)$:

$$T^{\Delta}_{\alpha\beta;l}(R) = \frac{1}{\pi}P_l^{(\beta)\Delta_\beta}\int_{-\infty}^{+\infty} \mathrm{d}_\lambda E_\lambda^{H_l^{(\beta)}}V_{(\beta)}(R)\Omega_{l-}^{(\alpha)\Delta_\alpha}\frac{\epsilon}{(H_l^{(\alpha)}-\lambda)^2+\epsilon^2}. \tag{3.11}$$

We now note that when the system of particles satisfies condition (3.A) the relations (3.10) are valid for the screened hamiltonian given by (2.2), ie $l = 2$. When condition (3.A) is not satisfied we must resort to the cut-off scattering theory whose screened hamiltonian is given by (2.1), ie $l = 1$, for which the intertwining properties (3.10) are valid.

In order to derive (3.1) we must express (3.8) with $T^{\Delta}_{\alpha\beta;l}(R)$ given by (3.11) in terms of momentum eigenfunctions. Since we are dealing with a scattering process which satisfies the condition (3.A) it is convenient to work with the screened theory whose hamiltonian is given by (2.2), ie $l = 2$. The expansion of the relation (3.8) in terms of momentum eigenfunctions $\Phi^{(\gamma)}_{p_\gamma,\omega_\gamma}$ can be performed in an analogous manner to the expansion of the T operator in terms of momentum eigenfunctions (Prugovečki 1973b) which yields with $\hat{h}_\gamma = (\Phi^{(\gamma)}_{p_\gamma,\omega_\gamma}|h\rangle$ the following equality:

$$\begin{aligned}
&\int_{\Delta_\beta}\mathrm{d}p_\beta\,\bar{\hat{f}}_\beta\int_{\substack{\Delta_\alpha\\ E^{(\beta)}=E^{(\alpha)\prime}}}\mathrm{d}p'_\alpha\,(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|T_{\alpha\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})\hat{g}'_\alpha\\
&\quad=\int_{\Delta_\beta}\mathrm{d}p_\beta\,\bar{\hat{f}}_\beta\int_{\substack{\Delta_\alpha\\ E^{(\beta)}=E^{(\alpha)\prime}}}\mathrm{d}p'_\alpha\,\frac{\delta_{\alpha\beta}}{2\pi\mathrm{i}}(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})_{E^{(\beta)}=E^{(\alpha)\prime}}\hat{g}'_\alpha\\
&\quad+\lim_{R\to+\infty}\int_{\Delta_\beta}\mathrm{d}p_\beta\,\bar{\hat{f}}_\beta\int_{\substack{\Delta_\alpha\\ E^{(\beta)}=E^{(\alpha)\prime}}}\mathrm{d}p'_\alpha\,\exp(\mathrm{i}\Lambda^{(\alpha)}(R)+\mathrm{i}\Lambda^{(\beta)}(R))\\
&\quad\times(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|V_{(\beta)}(R)\Omega^{(\alpha)}_{2-}(R)|\Phi^{(\alpha)}_{p_\alpha,\omega_\alpha})\hat{g}'_\alpha
\end{aligned} \tag{3.12}$$

where the expression $(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})_{E^{(\beta)}=E^{(\alpha)\prime}}$ is related to $(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})$ by

$$(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha}) = \delta(E^{(\beta)}-E^{(\alpha)\prime})(\Phi^{(\beta)}_{p_\beta,\omega_\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})_{E^{(\beta)}=E^{(\alpha)\prime}}.$$

We note that the restriction of the integration in (3.12) to the regions Δ_α and Δ_β is due to the following equality $\Omega^{(\gamma)\Delta_\gamma}_{\pm}\Phi^{(\gamma)}_{p_\gamma,\omega_\gamma} = \Omega^{(\gamma)}_{\pm}\chi_{\Delta_\gamma}(p_\gamma)\Phi^{(\gamma)}_{p_\gamma,\omega_\gamma}$. The relation (3.1) follows for all $p_\beta\in\Delta_\beta$ and $p_\alpha\in\Delta_\alpha$ from (3.12) under the assumption that the limit $R\to+\infty$ and the integrations can be interchanged.

In the case of collisions for which condition (3.A) is not satisfied we must resort to the cut-off theory for which all Coulomb potentials are screened (2.1). The channel hamiltonians $H_1^{(\beta)}$ and $H_1^{(\alpha)}$ are now dependent on the cut-off parameter R which implies that the eigenfunctions depend on R via the bound states. Thus expanding (3.8) with the cut-off T operator $T^{\Delta}_{\alpha\beta;1}(R)$ given by (3.11) in terms of R-dependent eigenfunctions

$\Phi^{(\gamma)}_{p_\gamma,\omega_\gamma}(R)$ we arrive at the following equality:

$$\int_{\Delta_\beta} \mathrm{d}p_\beta\, \bar{\bar{f}}_\beta \int_{\substack{\Delta_\alpha \\ E^{(\beta)}=E^{(\alpha)\prime}}} \mathrm{d}p'_\alpha\, (\Phi^{(\beta)}_{p_\beta,\omega_\beta}|T_{\alpha\beta}|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha})\hat{g}'_\alpha$$

$$= \lim_{R\to+\infty} \sum_{\omega} \int_{\Delta_\beta} \mathrm{d}p_\beta\, \bar{\bar{f}}_\beta(R) \int_{\substack{\Delta_\alpha \\ E^{(\beta)}(R)=E^{(\alpha)}(R)'}} \mathrm{d}p'_\alpha \exp(\mathrm{i}\Lambda^{(\alpha)}(R)+\mathrm{i}\Lambda^{(\beta)}(R))$$

$$\times (\Phi^{(\beta)}_{p_\beta,\omega_\beta}(R)|\, V_{(\beta)}(R)\Omega^{(\alpha)\Delta_\alpha}_{1-}(R)|\Phi^{(\alpha)}_{p'_\alpha,\omega'_\alpha}(R))\hat{g}'_\alpha(R). \tag{3.13}$$

Under the assumption that the limit $R \to +\infty$ and the integrations and summations can be interchanged and the further assumptions that the screened bound states with total energy $E_{(\gamma)}(R)$ converge pointwise as $R \to +\infty$ to the Coulomb-like bound states with energy $E_{(\gamma)}$ we arrive at the following relation:

$$\langle p_\beta, \omega_\beta|\, T_{\alpha\beta}|p'_\alpha, \omega'_\alpha\rangle_{E^{(\beta)}=E^{(\alpha)\prime}}$$

$$= \lim_{R\to+\infty} \exp(\mathrm{i}\Lambda^{(\alpha)}(R)+\mathrm{i}\Lambda^{(\beta)}(R))$$

$$\times \langle p_\beta, \omega_\beta; R|\, V_{(\beta)}(R)\Omega^{(\alpha)}_{1-}(R)|p'_\alpha, \omega'_\alpha; R\rangle_{E^{(\beta)}=E^{(\alpha)\prime}} \tag{3.14}$$

for all $p_\beta \in \Delta_\beta$ and $p_\alpha \in \Delta_\alpha$.

The assumption that the limit $R \to +\infty$ can be taken under the integral depends explicitly on the pointwise limit of the integrand existing. The following lemma will indicate that even for the two-body on-energy-shell screened T matrix the pointwise limit interpretation (3.1) is not valid.

Lemma 3.2. Suppose that for the case of two-body scattering the conditions (3.5) and (3.6) are satisfied and that the ranges of the renormalized wave operators are equal, ie $R_+ = R_-$. Then the restricted S operator $S^\Delta = \Omega^{\Delta *}_+\Omega^\Delta_-$ can be written as

$$S^\Delta = s - \lim_{R\to+\infty} \exp(\mathrm{i}\Lambda(R))\Omega^\Delta_+(R)^*\Omega^\Delta_-(R)\exp(\mathrm{i}\Lambda(R)) \tag{3.15}$$

and

$$P^\Delta = 2\pi\mathrm{i}\omega - \lim_{R\to+\infty} T^\Delta(R). \tag{3.16}$$

Proof. By lemma (2.2) we have

$$S^\Delta = \omega - \lim_{R\to+\infty} \exp(\mathrm{i}\Lambda(R))\Omega^\Delta_+(R)^*\Omega^\Delta_-(R)\exp(\mathrm{i}\Lambda(R)). \tag{3.17}$$

Since S^Δ is unitary the following inequality is valid for all $f \in L^2(\mathbb{R}^3)$

$$\|\exp(\mathrm{i}\Lambda(R))\Omega^\Delta_+(R)^*\Omega^\Delta_-(R)\exp(\mathrm{i}\Lambda(R))f\| \leqslant \|S^\Delta f\|$$

which together with (3.17) implies (3.15) (Prugovečki 1971a, lemma (6.2), p 334).

The screened T operator is given by

$$T^\Delta(R) = \frac{1}{2\pi\mathrm{i}}(\Omega^\Delta_-(R)^* - \Omega^\Delta_+(R)^*)\Omega^\Delta_-(R).$$

By (3.5) we have

$$\omega - \lim_{R\to+\infty} \Omega^\Delta_-(R)^*\Omega^\Delta_-(R) = P^\Delta.$$

Thus in order for (3.16) to hold we require that

$$\omega - \lim_{R\to+\infty} \Omega_+^{\Delta}(R)^*\Omega_-^{\Delta}(R) = 0.$$

This relation is a consequence of (3.15), (3.6) and the following equality:

$$\Omega_+^{\Delta}(R)^*\Omega_-^{\Delta}(R) = \exp(-2\mathrm{i}\Lambda(R))\exp(\mathrm{i}\Lambda(R))\Omega_+^{\Delta}(R)^*\Omega_-^{\Delta}(R)\exp(\mathrm{i}\Lambda(R)).$$

4. Conclusions

It must be emphasized that due to the Hilbert space approach adopted in this paper all relations involving eigenfunctions are valid only in the sense of their derivation from the corresponding operator relations. For example in the case of two-body Coulomb scattering the relation (3.16) when written in terms of momentum eigenfunctions implies that the pointwise limit relation (3.1) between the on-energy-shell Coulomb-like T matrix and the screened on-energy-shell Coulomb-like T matrix does not hold. Thus this relationship should be interpreted in the weak sense (3.12).

For the case of two-body Coulomb-like scattering the results of lemma (3.2) in conjunction with (3.12) seem to suggest that the screened on-energy-shell T matrix has the following form for large R:

$$\begin{aligned}\langle \boldsymbol{p}|V(R)\Omega_-(R)|\boldsymbol{p}'\rangle_{p^2/2m=p'^2/2m} &\simeq \exp(-2\mathrm{i}\Lambda(R))\langle \boldsymbol{p}|T|\boldsymbol{p}'\rangle_{p^2/2m=p'^2/2m} \\ &\quad + \frac{(2\pi)^2}{mp\mathrm{i}}[1-\exp(-2\mathrm{i}\Lambda(R))]\delta(\cos\theta-\cos\theta')\delta(\phi-\phi') \end{aligned} \qquad (4.1)$$

where m is the reduced mass, p, θ, ϕ (p', θ', ϕ') denotes the polar coordinates of the relative momentum $\boldsymbol{p}$ ($\boldsymbol{p}'$) and $\langle \boldsymbol{p}|T|\boldsymbol{p}'\rangle_{p^2/2m=p'^2/2m}$ denotes the on-energy-shell Coulomb T matrix. Much of the current work on the Coulomb T matrix (cf Chen and Chen 1971, Nuttall and Stagat 1971, and for a general survey with references to earlier results Chen and Chen 1972 and McDowell and Coleman 1970) has been concerned with understanding the energy-shell limit of the expression $\langle \boldsymbol{p}|V_C\Omega_-|\boldsymbol{p}'\rangle$. A calculation has been performed however for the cut-off Coulomb scattering amplitude (Rodberg and Thaler 1967, p 72, equation (5.50)). The results of this calculation can be shown to agree with the scattering amplitude obtained from (4.1) by the use of the relation

$$f_R(\theta) = -\frac{m}{2\pi}\langle \boldsymbol{p}|V(R)\Omega_-(R)|\boldsymbol{p}'\rangle.\dagger$$

It has been shown (Dollard 1968) that the screened approach is applicable to time-dependent scattering theory involving short-range potentials. In particular for the channel where all N particles are asymptotically free the screened short-range wave operators exist and converge to the short-range wave operators when the screening is removed. Expressing these time-dependent results in terms of eigenfunctions one can conclude that the screened short-range distorted waves and T matrix converge to the corresponding distorted waves and T matrix of the short-range stationary scattering theory when the screening is removed.

† We note that a factor of $1/k$ is missing from the second term of equation (5.50) of Rodberg and Thaler (1967).

The cut-off approach can also be used to study long-range potentials other than the Coulomb potential. In the case of N particle scattering not involving bound states with the particles interacting via potentials having the asymptotic form $r^{-l}, \frac{3}{4} \leqslant l < 1$, analogous results to those proven in this paper can be shown to hold (Zorbas 1974). For such potentials the function $\Lambda^{(\gamma)}(R)$ takes the following form:

$$(\Lambda^{(\gamma)}(R)\Psi)(p_\gamma, \omega_\gamma) = \sum_{j<k} \frac{C_{jk}M_j^l M_k^l}{|M_j \boldsymbol{p}_k - M_k \boldsymbol{p}_j|^l} \left[\int_{t_0}^{+\infty} \mathrm{d}s\, s^{-l} g_{jk}^R \left(\frac{s}{M_j M_k} |M_j \boldsymbol{p}_k - M_k \boldsymbol{p}_j| \right) + \frac{t_0^{1-l}}{1-l} \right] \Psi(p_\gamma, \omega_\gamma) \tag{4.2}$$

where C_{jk} are real constants.

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